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Quasiparticle Pairs in Boltzmann Kinetic Theory

V.L. Saveliev

Institute of Ionosphere, NCSRT, Almaty, Kamenskoe Plato, 050020, Kazakhstan

Parameterization of collisions by rotation matrix

In order to construct the collision integral in the Boltzmann kinetic equation, a collision of two particles is usually determined by setting a direction $\boldsymbol{n}$, $\left(n^2 = 1\right)$ of the post-collision relative velocity [1]:

$$\begin{aligned} \boldsymbol{v}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} + \left|\boldsymbol{v} - \boldsymbol{u}\right|\boldsymbol{n}}{1+m}, \\ \boldsymbol{u}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} - m\left|\boldsymbol{v} - \boldsymbol{u}\right|\boldsymbol{n}}{1+m}, \end{aligned} \tag{1}$$

where $m = \dfrac{m_1}{m_2}$. The collision integral is given accordingly by:

$$I(f,\psi) = \int \mathrm{v}\, \sigma_\theta(\mathrm{v}^2, \frac{\mathbf{v}\cdot\boldsymbol{n}}{\mathrm{v}})\left[\, f(\boldsymbol{v}')\psi(\boldsymbol{u}') - f(\boldsymbol{v})\psi(\boldsymbol{u})\right] d\Omega_{\boldsymbol{n}} d\boldsymbol{u}\,, \tag{2}$$

where $\mathbf{v} = \boldsymbol{v} - \boldsymbol{u}$ and σ_θ is the differential collision cross section.
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[1] L.D. Landau and E.M. Lifshitz, *Mechanics*, 1976.

In our approach [2,3], we proposed to parameterize a collision by a rotation matrix $\hat{R} \in O_3^+$. In this case, the transformation of the velocities due to collision becomes a linear one:

$$\begin{aligned} \boldsymbol{v}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} + \hat{R}(\boldsymbol{v} - \boldsymbol{u})}{1+m}, \\ \boldsymbol{u}' &= \frac{m\boldsymbol{v} + \boldsymbol{u} - m\hat{R}(\boldsymbol{v} - \boldsymbol{u})}{1+m}. \end{aligned} \tag{3}$$

Rotation on the angle ϕ around the axis $\boldsymbol{n}$ is given by the following formula [*Hamermesh*]:

$$\hat{R} = e^{\phi \hat{n}} = \left(1 - \cos\phi\right)\hat{n}^2 + \left(\sin\phi\right)\hat{n} + 1, \tag{4}$$

$$\hat{n} \stackrel{df}{=} \boldsymbol{n}\times, \qquad \hat{n}\,\mathbf{v} = \boldsymbol{n}\times\mathbf{v}. \tag{5}$$

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[2]Saveliev, V.L., RGD 22, AIP Conf. Proc. **585**, 101, (2001),
[3] Saveliev,V.L., and Nanbu K., Phys. Rev., E 65, 051205, pp.1-9, (2002)

The velocities $\boldsymbol{v}'$ and $\boldsymbol{u}'$ are determined by a partitioned (2×2 cells) scattering matrix $\hat{S}$. The size of each cell is obviously (3×3):

$$\xi' = \hat{S}\xi, \qquad \xi = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{pmatrix}, \qquad \hat{S} = \begin{pmatrix} \dfrac{m+\hat{R}}{1+m} & \dfrac{1-\hat{R}}{1+m} \\ \dfrac{m(1-\hat{R})}{1+m} & \dfrac{1+m\hat{R}}{1+m} \end{pmatrix}, \tag{6}$$

ξ is a 6-dimensional bivector consisting of the components $\boldsymbol{v}$ and $\boldsymbol{u}$.

The scattering matrixes $\hat{S}(\hat{R})\left(\hat{R} \in O_3\right)$ constitute a group that is isomorphic to the group of orthogonal matrixes O_3:

$$\hat{S}(\hat{R}_1) \cdot \hat{S}(\hat{R}_2) = \hat{S}(\hat{R}_1 \cdot \hat{R}_2), \quad \hat{S}^{-1}(\hat{R}) = \hat{S}(\hat{R}^{-1}), \quad \left|\det \hat{S}\right| = 1, \tag{7}$$

$$d\boldsymbol{v}d\boldsymbol{u} = d\boldsymbol{v}'d\boldsymbol{u}' \tag{8}$$

Relation (8) is simpler than $d\boldsymbol{v}d\boldsymbol{u}d\Omega' = d\boldsymbol{v}'d\boldsymbol{u}'d\Omega$ for the conventional [4] parameterization.

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[4] Ferziger,J. H., and Kaper,H. G., Mathematical Theory of Transport Processes in Gases, 1972.

Boltzmann Collision Integral

To rewrite the collision integral (2), the averaging over directions of postcolisional relative velocity $\mathbf{v}'$ should be replaced by averaging over the group O_3^+ $(\dfrac{d\Omega_{\mathbf{v}'}}{4\pi} \to \dfrac{d\hat{R}}{8\pi^2})$:

$$d\hat{R} = d\hat{R}_0\hat{R} = d\hat{R}\hat{R}_0 = d\hat{R}^{-1}, \qquad d\hat{R} = 2(1-\cos\phi)d\phi d\Omega_{\boldsymbol{n}}, \quad \int d\hat{R} = 8\pi^2 \tag{9}$$

Transformations of velocities induce transformations of functions on bivector $(\boldsymbol{v}, \boldsymbol{u})$:

$$e^{-\phi\hat{\sigma}} f(\boldsymbol{v})\psi(\boldsymbol{u}) = f(\boldsymbol{v}')\psi(\boldsymbol{u}') \tag{10}$$

$$\hat{\sigma} = \boldsymbol{n}\cdot\hat{\boldsymbol{\sigma}}, \qquad \hat{\boldsymbol{\sigma}} = \frac{\partial}{\partial\mathbf{v}}\times\mathbf{v} = -\mathbf{v}\times\frac{\partial}{\partial\mathbf{v}} = -\frac{1}{1+m}\mathbf{v}\times\left(\frac{\partial}{\partial\boldsymbol{v}} - m\frac{\partial}{\partial\boldsymbol{u}}\right), \quad \mathbf{v} = \boldsymbol{v}-\boldsymbol{u} \tag{11}$$

The Boltzmann collision integral expressed via scattering operator reads:

$$I(f,\psi) = \int b\big(\mathrm{v},\mu\big)\big[f(\boldsymbol{v}')\psi(\boldsymbol{u}') - f(\boldsymbol{v})\psi(\boldsymbol{u})\big]\frac{d\hat{R}}{2\pi}d\boldsymbol{u} = \int d\boldsymbol{u}\,\hat{\chi}f(\boldsymbol{v})\psi(\boldsymbol{u}) \tag{12}$$

Scattering operator in prefix form

$$\hat{\chi} = \int \frac{d\hat{R}}{2\pi} b\left(\mathrm{v}, \mu\right)\left[e^{-\phi\hat{\sigma}} - 1\right]. \tag{13}$$

A rather simple generator of rotations $\hat{\sigma}$ determines all general properties of the scattering operator $\hat{\chi}$ and accordantly of the Boltzmann collision integral:

$$\hat{\sigma} = \boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}} \tag{14}$$

$$\hat{\boldsymbol{\sigma}} = -\mathbf{v} \times \frac{\partial}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \times = -\frac{1}{1+m} \mathbf{v} \times \left(\frac{\partial}{\partial \boldsymbol{v}} - m \frac{\partial}{\partial \boldsymbol{u}}\right) \tag{15}$$

- The product of Maxwellians is an eigenfunction of the generator $\hat{\sigma}$:

$$\hat{\sigma}\, e^{-\frac{m_1 v^2}{2kT}}\, e^{-\frac{m_2 u^2}{2kT}} = 0 \tag{16}$$

- It commutates with three invariants and obviously with Casimir operator $\hat{\boldsymbol{\sigma}}^2$:

$$\left[\hat{\sigma}, \mathrm{v}^2\right] = 0, \qquad \left[\hat{\sigma}, \mu\right] = 0, \qquad \left[\hat{\sigma}, \left(m\boldsymbol{v} + \boldsymbol{u}\right)\right] = 0, \quad \left[\hat{\sigma}, \hat{\boldsymbol{\sigma}}^2\right] = 0. \tag{17}$$

Parameterisation of Collisions by Rotations on π angles

If we parameterize the collision of two particles by the rotation matrix $e^{\pi\hat{n}}$, which is a rotation of the relative velocity vector $\mathbf{v}$ on the π angle around the axis $\boldsymbol{n}$,

$$\begin{aligned}\mathbf{v}' &= e^{\pi\hat{n}}\mathbf{v} = \Big[\big(1-\cos\pi\big)\hat{n}^2 + \big(\sin\pi\big)\hat{n} + 1\Big]\mathbf{v} \\ &= \Big[2\boldsymbol{n}\times\boldsymbol{n}\times +1\Big]\mathbf{v} = 2\boldsymbol{n}\,\boldsymbol{n}\cdot\mathbf{v} - \mathbf{v},\end{aligned} \tag{18}$$

then the Boltzmann collision integral takes the form:

$$\begin{aligned}I(f,\psi) &= \int d\boldsymbol{u}d\Omega_n 2\big|\cos\theta\big| b\big(\mathrm{v},\mu\big)\big[f(\boldsymbol{v}')\psi(\boldsymbol{u}') - f(\boldsymbol{v})\psi(\boldsymbol{u})\big] \\ &= \int d\boldsymbol{u}\,\hat{\chi}\,f(\boldsymbol{v})\psi(\boldsymbol{u}),\end{aligned} \tag{19}$$

where the scattering operator $\hat{\chi}$ is the operator of averaging over the all directions of axis $\boldsymbol{n}$:

$$\hat{\chi} = 2\int d\Omega_n \big|\cos\theta\big|\, b\big(\mathrm{v},\mu\big)\Big(e^{\pi\hat{\sigma}} - 1\Big), \tag{20}$$

where $\cos\theta = \dfrac{\boldsymbol{n}\cdot\mathbf{v}}{\mathrm{v}}, \qquad \mu = \dfrac{\mathbf{v}'\cdot\mathbf{v}}{\mathrm{v}^2} = \cos 2\theta\,.$

Renormalization of the Scattering Operator and Collision Integral

To renormalize $\hat{\chi}$ we expand operator $e^{\pi\hat{\sigma}}$ in the Taylor series with a residual term:

$$e^{\pi\hat{\sigma}} = 1 + \pi\hat{\sigma} ... + \frac{1}{(n-1)!}(\pi\hat{\sigma})^{n-1} + \frac{1}{n!}(\pi\hat{\sigma})^{n} \int_0^1 d\alpha\, q_n(\alpha) e^{\alpha\pi\hat{\sigma}}, \tag{21}$$

where $q_n(\alpha) = n(1-\alpha)^{n-1}$, $\int_0^1 d\alpha\, q_n(\alpha) = 1.$

$$\left(e^{\pi\hat{\sigma}} - 1\right) = \sum_{k=1}^{2k<n} \frac{(\pi\hat{\sigma})^{2k}}{(2k)!} + \frac{(\pi\hat{\sigma})^{n}}{n!} \int_0^1 d\alpha\, q_n(\alpha) \frac{1}{2}\left[e^{\alpha\pi\hat{\sigma}} + (-1)^n e^{-\alpha\pi\hat{\sigma}}\right] \tag{22}$$

Scattering operator $\hat{\chi}$ for any given $n > 0$ has the exact expression:

$$\hat{\chi} = \sum_{k=1}^{2k<n} \frac{\pi^{2k}}{(2k)!} \left\langle n_{i_1} ... n_{i_{2k}} \right\rangle \hat{\sigma}_{i_1} ... \hat{\sigma}_{i_{2k}} + \frac{\pi^n}{n!} \left\langle \frac{\hat{\sigma}^n}{2}\left[e^{\alpha\pi\hat{\sigma}} + (-1)^n e^{-\alpha\pi\hat{\sigma}}\right] \right\rangle \tag{23}$$

$$\langle [...] \rangle = \int d\alpha\, q_n(\alpha) d\Omega_n\, 2\left|\cos\theta\right| b(\mathrm{v}, \mu)[...], \quad \mu = \cos 2\theta, \quad d\Omega_{\boldsymbol{n}} = \sin\theta d\theta d\varphi \tag{24}$$

Case $n = 2,\ q_2(\alpha) = 2(1-\alpha)$.

Therefore, for the scattering operator we have the following:

$$\hat{\chi} = \frac{\pi^2}{2}\left\langle \frac{\hat{\sigma}^2}{2}\left[e^{\alpha\pi\hat{\sigma}} + e^{-\alpha\pi\hat{\sigma}} \right] \right\rangle_2$$
$$= \frac{\pi^2}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \int d\alpha\, 2\left(1-\alpha\right) d\Omega_n\, 2\left|\cos\theta\right|\, b\left(\mathrm{v},\mu\right) \frac{[\mathbf{v}\times \boldsymbol{n}]\hat{\sigma}}{2}\left[e^{\alpha\pi\hat{\sigma}} + e^{-\alpha\pi\hat{\sigma}} \right] \tag{25}$$

The Boltzmann collision integral reads:

$$I\left(f,\psi\right) = \int d\boldsymbol{u}\hat{\chi} f\left(\boldsymbol{v}\right)\psi\left(\boldsymbol{u}\right) = -\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{J}\left(f,\psi\right), \tag{26}$$

where

$$\boldsymbol{J}\left(f,\psi\right) = -\frac{\pi^2}{4\left(1+m\right)^2}\int d\boldsymbol{u}\left\langle \left(\boldsymbol{n}\times\mathbf{v}\right)\left(\boldsymbol{n}\times\mathbf{v}\right)\cdot\left(\frac{\partial}{\partial\boldsymbol{v}} - m\frac{\partial}{\partial\boldsymbol{u}}\right)\left[f'_+\psi'_+ + f'_-\psi'_-\right]\right\rangle_2 , \tag{27}$$

$$\left\langle \dots \right\rangle_2 = \int d\alpha\, 2\left(1-\alpha\right) d\Omega_n\, 2\left|\cos\theta\right|\, b\left(\mathrm{v},\mu\right)[\dots],$$
$$f'_\pm = f\left(\boldsymbol{v}'_\pm\right),\quad \psi'_\pm = \psi\left(\boldsymbol{u}'_\pm\right),\quad \cos\theta = (\boldsymbol{n}\cdot\mathbf{v})\,/\,\mathrm{v},\quad \mu = (\mathbf{v}'\cdot\mathbf{v})\,/\,\mathrm{v}^2 = \cos 2\theta. \tag{28}$$

Quasiparticles

The Boltzmann equation for molecules

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{e_1}{m_1}\boldsymbol{E}+\frac{e_1}{m_1 c}\boldsymbol{v}\times\boldsymbol{B}+\boldsymbol{g}\right)f = I(f,f) \qquad (29)$$

can be rewritten in the form of the Liouville equation :

$$\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{e_1}{m_1}\boldsymbol{E}+\frac{e_1}{m_1 c}\boldsymbol{v}\times\boldsymbol{B}+\boldsymbol{g}+\frac{1}{m_1}\boldsymbol{F_{coll}}\right)f = 0, \qquad (30)$$

where, the effect of collisions accounted by nonlocal kinetic force $\boldsymbol{F}_{coll} = m_1\boldsymbol{J} \,/\, f(\boldsymbol{v})$, which depends on the distribution functions $f(\boldsymbol{v})$.

- Equation (30) allows us to consider the distribution function $f(\boldsymbol{v},\boldsymbol{r},t)$ as a density of quasiparticles, which are moving along smooth trajectories under the influence of the nonlocal force $\boldsymbol{F}_{coll}$. Quasiparticles do not jump in velocity space as it was in the case of the classical Boltzmann equation. This equation provides new opportunities for numerical simulation of gas flows.
- *Important physical conclusion:*
  It is possible considerably simplify a microdynamics of the system while the evolution of a distribution function remains unchanged.

Pairs of Quasiparticles: Motivation and Outline

• Pair collision is the main interaction process in the Boltzmann gas dynamics. To take account of this interaction one need to possess a two-particle distribution function. In our report we sagest the kinetic equation for two-particle distribution function in the gas mixtures that is written in frame of exactly the same physical assumptions as was adopted by Ludwig Boltzmann when he wrote his famous equation.

• Our equation contains the linear scattering operator in its right part which is simpler then the collision integral in the corresponding Boltzmann equation.

• We present factorization of the scattering operator.

• Renormalized forms of the scattering operator allow us to substitute real molecules by pairs of quasiparticles, which don't jump in velocity space but rotates smoothly with an angular velocity, which depends on distribution function

• Two particle approach provides fulfillment of conservation laws in numerics automatically.

Equation for two-particles Distribution Functions in Mixture of Boltzmann Gases

We proposed the set of equations for two-particle distribution functions $F_{\alpha\beta}(\boldsymbol{v}_1, \boldsymbol{r}_1, \boldsymbol{v}_2, \boldsymbol{r}_2, t)$ as a mathematical model for gas mixture consisting of k components when two particles collisions are overwhelming in the system. Indexes α and β label components in the interval from 1 to k. Here $\boldsymbol{v}_1$ and $\boldsymbol{r}_1$ are velocity and position of the particle from α-component; $\boldsymbol{v}_2, \boldsymbol{r}_2$ are velocity and position of the particle from β-component

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}_1} \cdot \boldsymbol{v}_1 + \frac{\partial}{\partial \boldsymbol{r}_2} \cdot \boldsymbol{v}_2 + \frac{\partial}{\partial \boldsymbol{v}_1} \cdot \boldsymbol{a}_\alpha + \frac{\partial}{\partial \boldsymbol{v}_2} \cdot \boldsymbol{a}_\beta \right) F_{\alpha\beta}\left(\boldsymbol{v}_1, \boldsymbol{r}_1, \boldsymbol{v}_2, \boldsymbol{r}_2, t\right)$$

$$= N\delta\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right) \int \frac{d\hat{R}}{2\pi} b_{\alpha\beta}\left(\mu, \mathrm{v}\right) \left[f_\alpha\left(\boldsymbol{v}_1'\right) f_\beta\left(\boldsymbol{v}_2'\right) - f_\alpha\left(\boldsymbol{v}_1\right) f_\beta\left(\boldsymbol{v}_2\right)\right], \tag{31}$$

.

$$f_\alpha\left(\boldsymbol{v}_1, \boldsymbol{r}_1, t\right) = \frac{1}{2N} \sum_{\beta=1}^{K} \int d\boldsymbol{r}_2 d\boldsymbol{v}_2 \left[F_{\alpha\beta}\left(\boldsymbol{v}_1, \boldsymbol{r}_1, \boldsymbol{v}_2, \boldsymbol{r}_2, t\right) + F_{\beta\alpha}\left(\boldsymbol{v}_2, \boldsymbol{r}_2, \boldsymbol{v}_1, \boldsymbol{r}_1, t\right)\right].$$

Scattering Operator for Gas Mixture

Instead of collision integral for the Boltzmann equation, in the two particle equation we have the scattering operator $\hat{\chi}_{\alpha\beta}$ to account collisions:

$$\hat{\chi}_{\alpha\beta} f_\alpha(\boldsymbol{v}_1) f_\beta(\boldsymbol{v}_2) = \int \frac{d\hat{R}}{2\pi} b_{\alpha\beta}(\mu, \mathrm{v}) \left[f_\alpha(\boldsymbol{v}_1') f_\beta(\boldsymbol{v}_2') - f_\alpha(\boldsymbol{v}_1) f_\beta(\boldsymbol{v}_2) \right], \tag{32}$$

$$\boldsymbol{v}_1' = \frac{m\boldsymbol{v}_1 + \boldsymbol{v}_2 + \hat{R}(\boldsymbol{v}_1 - \boldsymbol{v}_2)}{1+m}, \quad \boldsymbol{v}_2' = \frac{m^{-1}\boldsymbol{v}_1 + \boldsymbol{v}_2 + \hat{R}(\boldsymbol{v}_2 - \boldsymbol{v}_1)}{1+m^{-1}}, \quad m = \frac{m_\alpha}{m_\beta},$$
$$b_{\alpha\beta}(\mu, \mathrm{v}) = b_{\beta,\alpha}(\mu, \mathrm{v}) = \mathrm{v}\frac{d\sigma_{\alpha\beta}}{d\Omega}, \quad \mathbf{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2, \quad \mu = \frac{\mathbf{v}\cdot\hat{R}\mathbf{v}}{\mathrm{v}^2}, \tag{33}$$

$$\hat{R} = e^{\phi\hat{n}} = (1 - \cos\phi)\hat{n}^2 + (\sin\phi)\hat{n} + 1, \qquad \hat{n}\,\mathbf{v} = \boldsymbol{n} \times \mathbf{v}. \tag{34}$$

$\hat{R}$ is a rotation matrix, .., $d\Omega_{\boldsymbol{n}} = \sin\theta d\theta d\varphi$, $0 \le \phi, \theta \le \pi$, $0 \le \varphi \le 2\pi$; $\boldsymbol{n}$ is an axis and ϕ is an angle of the rotation, $\sigma_{\alpha\beta}$ is a cross section for particles from α-component colliding with particles from β-component.

Normalizations:

$$\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right) = N_\alpha N_\beta, \qquad \sum_{\alpha=1}^{k} N_\alpha = N, \quad \alpha,\beta = 1,\ldots,k. \quad (35)$$

N_α is a number of particles in the α–component of the mixture. N is a total number of particles.

All other normalizations follow from eq.(35):

$$\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right) = N_\alpha, \quad \int d\boldsymbol{v}_1 f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right) = n_\alpha\left(\boldsymbol{r}_1,t\right), \quad \int d\boldsymbol{r}_1 n_\alpha\left(\boldsymbol{r}_1\right) = N_\alpha. \quad (36)$$

Symmetric Solutions

Eq. (31) has symmetric solutions and we will consider only them hereinafter:

$$F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right) = F_{\beta\alpha}\left(\boldsymbol{v}_2,\boldsymbol{r}_2,\boldsymbol{v}_1,\boldsymbol{r}_1,t\right). \quad (37)$$

Boltzmann Equation follows directly from Two-particle equation

Boltzmann equation for one-particle distribution function f_α follows from eq. (31) without any additional assumptions after a simple integration over velocities and the positions of the second particle. Also to do this, one need to use an invariance of eq.(31) under the simultaneous interchange of indexes: $1 \rightleftarrows 2$ and $\alpha \rightleftarrows \beta$:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}_1} \cdot \boldsymbol{v}_1 + \frac{\partial}{\partial \boldsymbol{v}_1} \cdot \boldsymbol{a}_\alpha\right) f_\alpha = \sum_{\beta=1}^{k} \int d\boldsymbol{v}_2 \hat{\chi}_{\alpha\beta} f_\alpha\left(\boldsymbol{v}_1\right) f_\beta\left(\boldsymbol{v}_2\right). \tag{38}$$

Chaos Projector

The product $f_\alpha f_\beta$ can be obtained from $F_{\alpha\beta}$ by the chaos projector:

$$f_\alpha\left(\boldsymbol{v}_1,\boldsymbol{r}_1,t\right)\cdot f_\beta\left(\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)=\left[\hat{P}F_{\alpha\beta}\right]\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right), \tag{39}$$

$$\hat{P}F_{\alpha,\beta}=\frac{\left[\sum_{\beta=1}^{k}\int d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)\right]\left[\sum_{\alpha=1}^{k}\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)\right]}{\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k}\int d\boldsymbol{r}_1 d\boldsymbol{v}_1 d\boldsymbol{r}_2 d\boldsymbol{v}_2 F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)}, \tag{40}$$

$$\hat{P}^2F=\hat{P}F,\qquad \hat{P}cF=c\hat{P}F,\qquad c-const, \tag{41}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{r}_1}\cdot\boldsymbol{v}_1+\frac{\partial}{\partial \boldsymbol{r}_2}\cdot\boldsymbol{v}_2+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}_\alpha+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}_\beta\right)F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right)$$
$$=N\,\delta\left(\boldsymbol{r}_1-\boldsymbol{r}_2\right)\hat{\chi}_{\alpha\beta}\hat{P}F_{\alpha\beta}\left(\boldsymbol{v}_1,\boldsymbol{r}_1,\boldsymbol{v}_2,\boldsymbol{r}_2,t\right). \tag{42}$$

Homogeneous Simple Gas

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \boldsymbol{v}_1}\cdot\boldsymbol{a}+\frac{\partial}{\partial \boldsymbol{v}_2}\cdot\boldsymbol{a}\right)F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,t\right)=n\,\hat{\chi}f\left(\boldsymbol{v}_1\right)f\left(\boldsymbol{v}_2\right),$$
$$f\left(\boldsymbol{v},t\right)=\frac{1}{2n}\int d\boldsymbol{v}_2\left[F\left(\boldsymbol{v},\boldsymbol{v}_2,t\right)+F\left(\boldsymbol{v}_2,\boldsymbol{v},t\right)\right]. \tag{43}$$

$$F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,t\right)=F\left(\boldsymbol{v}_2,\boldsymbol{v}_1,t\right),\quad \int d\boldsymbol{v}_1 d\boldsymbol{v}_2 F\left(\boldsymbol{v}_1,\boldsymbol{v}_2,t\right)=n^2,\quad \int d\boldsymbol{v}\,f\left(\boldsymbol{v},t\right)=n \tag{44}$$

Invariance of Scattering Operator under Rotations

- Symmetry under the group of rotations is the main property of the scattering operator.

$$e^{\phi\hat{\sigma}}\hat{\chi}e^{-\phi\hat{\sigma}} = \hat{\chi}. \tag{45}$$

- Due to this invariance, Legendre polynomials, constructed with the help of arbitrary vector $\mathbf{v}_0$, are its eigenfunctions:

$$\hat{\chi}P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right) = \lambda_l P_l\left(\frac{\mathbf{v}\cdot\mathbf{v}_0}{\mathrm{v}\,\mathrm{v}_0}\right), \quad \lambda_l = \left[\hat{\chi}P_l\right]_{\mathbf{v}=\mathbf{v}_0} = 2\pi\int_{-1}^{1} d\mu\, b\left(\mathrm{v},\mu\right)\left[P_l\left(\mu\right)-1\right]. \tag{46}$$

- Using properties (45) and (46), it is possible to represent the scattering operator $\hat{\chi}$ in various equivalent forms:.

Factorization of Scattering Operator $\hat{\chi}$

Factorization of eigenvalues corresponds to factorization of the scattering operator $\hat{\chi}$:

$$\hat{\chi} = \hat{\boldsymbol{\sigma}}^2 \hat{\chi}_- \tag{47}$$

where

$$\hat{\chi}\_F(\mathbf{v}) = \int b\_(\mu) F(\mathbf{v}') d\Omega_{\mathbf{v}'},$$

$$b\_(\mu) = \frac{1}{2}\int_{-1}^{\mu} d\mu_2\, b\left(\mu_2\right) \ln \frac{\left(1+\mu\right)\left(1-\mu_2\right)}{\left(1-\mu\right)\left(1+\mu_2\right)}, \quad \int_{-1}^{1} b\_\left(\mu\right) d\mu = \int_{-1}^{1} \ln\left(\frac{2}{1+\mu}\right) b\left(\mu\right) d\mu\,, \tag{48}$$

Landau-Fokker-Plank Like Form of Boltzmann Collision Integral

Keeping in mind that

$$\hat{\boldsymbol{\sigma}} = \frac{\partial}{\partial \mathbf{v}} \times \mathbf{v}, \quad \hat{\boldsymbol{\sigma}}^2 = \frac{\partial}{\partial \mathbf{v}}\left(\mathrm{v}^2 - \cdot\mathbf{v}\mathbf{v}\cdot\right)\frac{\partial}{\partial \mathbf{v}},$$
$$\mathbf{v} = \boldsymbol{v} - \boldsymbol{u}, \quad \mathbf{w} = \frac{\boldsymbol{v} + m\boldsymbol{u}}{1+m}, \quad \frac{\partial}{\partial \mathbf{v}} = \frac{1}{1+m}\left(\frac{\partial}{\partial \boldsymbol{v}} - m\frac{\partial}{\partial \boldsymbol{u}}\right), \tag{49}$$

we exactly transform Boltzmann collision integral to the following Landau-Fokker-Plank like form:

$$I(f,\psi) = \frac{1}{\left(1+m\right)^2}\frac{\partial}{\partial v_i}\int d\boldsymbol{u}\left(\mathrm{v}^2\,\delta_{ik} - \mathrm{v}_i\,\mathrm{v}_k\right)\left(\frac{\partial}{\partial v_k} - m\frac{\partial}{\partial u_k}\right)\hat{\chi}_{-} f\left(\boldsymbol{v}\right)\psi\left(\boldsymbol{u}\right). \tag{50}$$

For "gracing" collisions $\hat{\chi}_{-} \approx \frac{1}{2}\left\langle 1-\mu\right\rangle$, where $\left\langle 1-\mu\right\rangle = 2\pi\int\limits_{-1}^{1} b\left(\mu\right)\left(1-\mu\right)d\mu$, and exact equation (50) reduces to approximate Landau-Fokker-Plank collision integral.

Example 2:

$$\hat{\chi} = -\frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{v} \times \hat{\boldsymbol{\Omega}} \right], \tag{51}$$

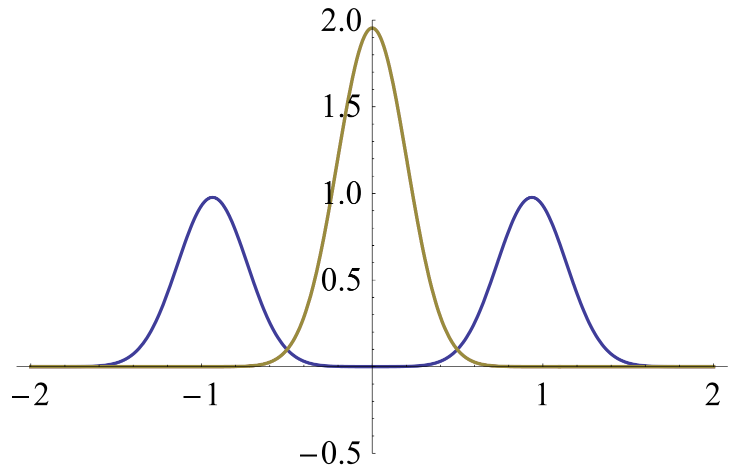
$$\hat{\boldsymbol{\Omega}} = \int d\phi \, d\Omega_n \, \frac{\mathbf{v} \times \mathbf{v} \times \boldsymbol{n}}{\mathrm{v}^2} \left| \cos\theta \right| b\left(\mathrm{v}, \cos 2\theta \right) \left[e^{\phi\hat{\sigma}} - e^{-\phi\hat{\sigma}} \right], \tag{52}$$

Form (51) allows us to consider the dynamics due to collisions as a rotation of the relative velocity $\mathbf{v}$ in the quasiparticle pair with the angular velocity $\boldsymbol{\Omega}$:

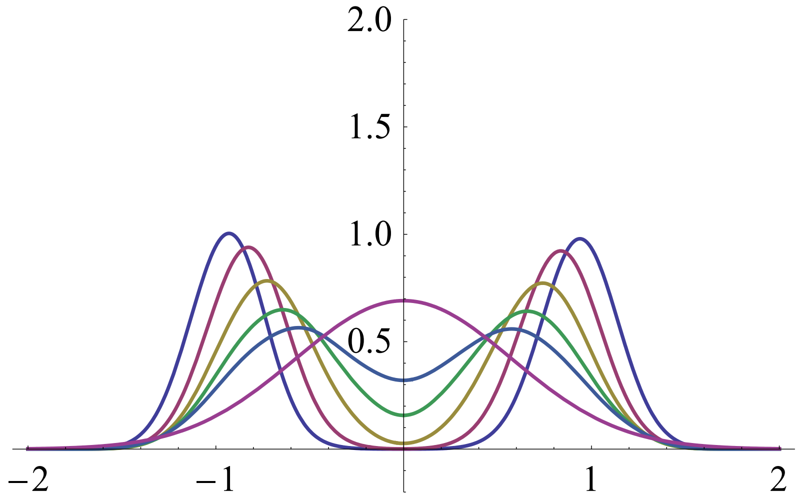
$$\boldsymbol{\Omega} = -\left[f_\alpha\left(\boldsymbol{v}\right) f_\beta\left(\boldsymbol{u}\right) \right]^{-1} \hat{\boldsymbol{\Omega}} \, f_\alpha\left(\boldsymbol{v}\right) f_\beta\left(\boldsymbol{u}\right) \tag{53}$$

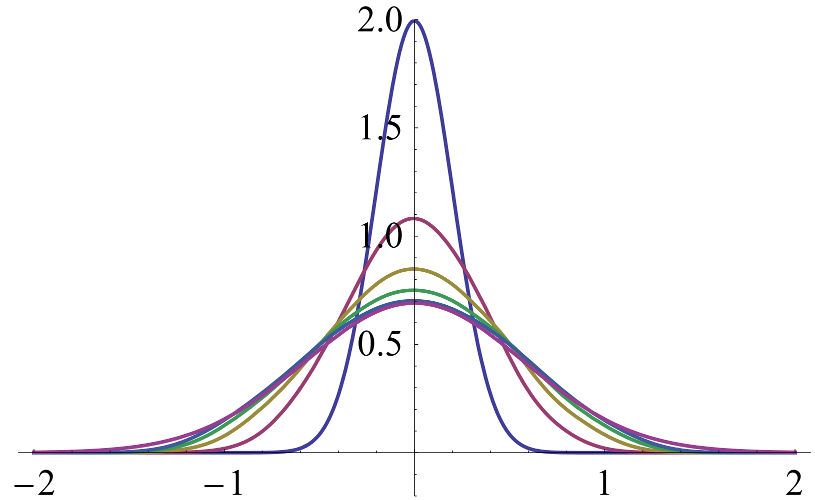
$$\begin{aligned} \boldsymbol{v}_1\left(t + dt\right) &= \boldsymbol{v}_1 + \frac{1}{1+m}\left[\left(1 - \cos(\Omega\, dt)\right) \frac{\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times}{\Omega^2} + \sin(\Omega\, dt) \, \frac{\boldsymbol{\Omega} \times}{\Omega} \right] \left[\boldsymbol{v}_1 - \boldsymbol{v}_2 \right], \\ \boldsymbol{v}_2\left(t + dt\right) &= \boldsymbol{v}_2 - \frac{1}{1+m}\left[\left(1 - \cos(\Omega\, dt)\right) \frac{\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times}{\Omega^2} + \sin(\Omega\, dt) \, \frac{\boldsymbol{\Omega} \times}{\Omega} \right] \left[\boldsymbol{v}_1 - \boldsymbol{v}_2) \right], \end{aligned} \tag{54}$$

$$\boldsymbol{r}_1\left(t + dt\right) = \boldsymbol{r}_1 + \boldsymbol{v}_1 dt, \qquad \boldsymbol{r}_2\left(t + dt\right) = \boldsymbol{r}_2 + \boldsymbol{v}_2 dt \,. \tag{55}$$



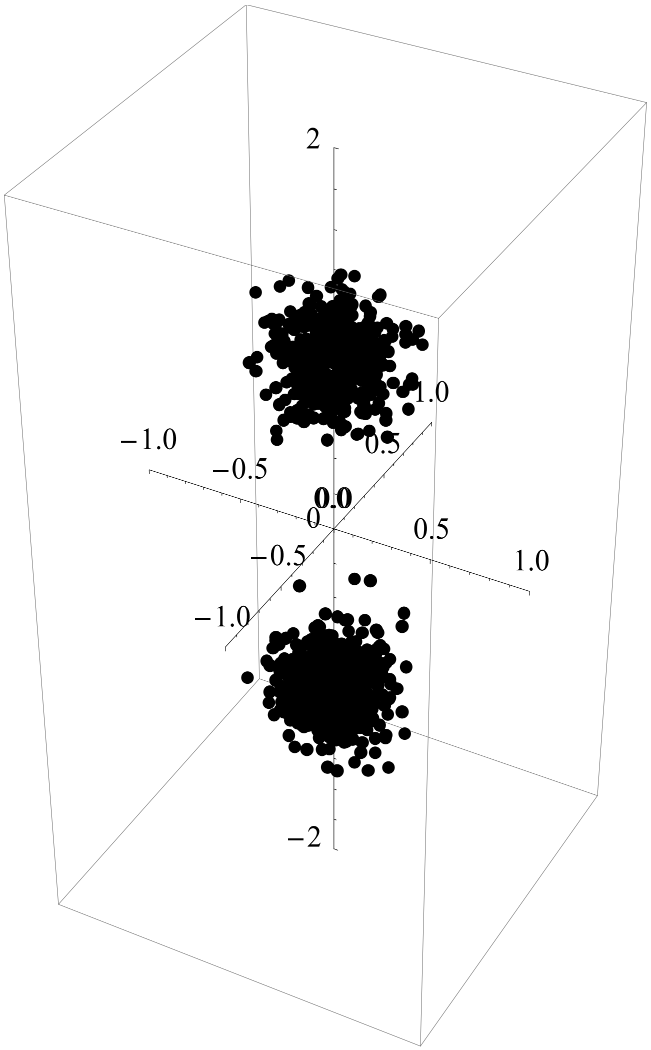
Initial distribution functions: $f(v_z)$– blue ; $f(v_x)=f(v_y)$– brown



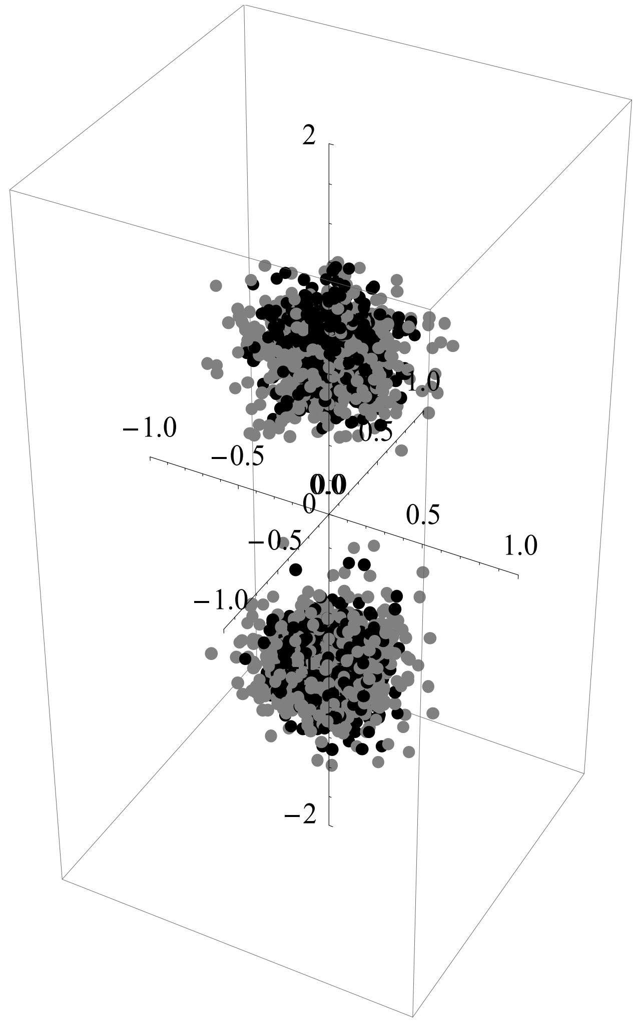


Evolution of distribution functions $f(v_z)$ and $f(v_x)$ to Maxwellian: $t = 0, 1, 2, 3, 4$

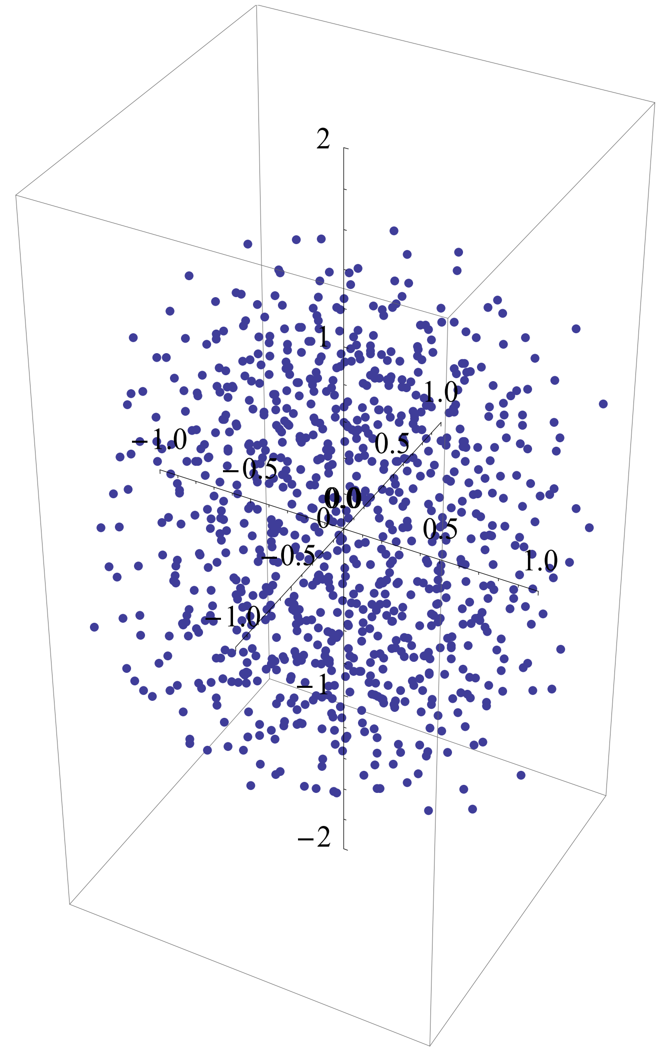
Velocities of 1000 quasiparticles in 3D velocity space



$t = 0;$

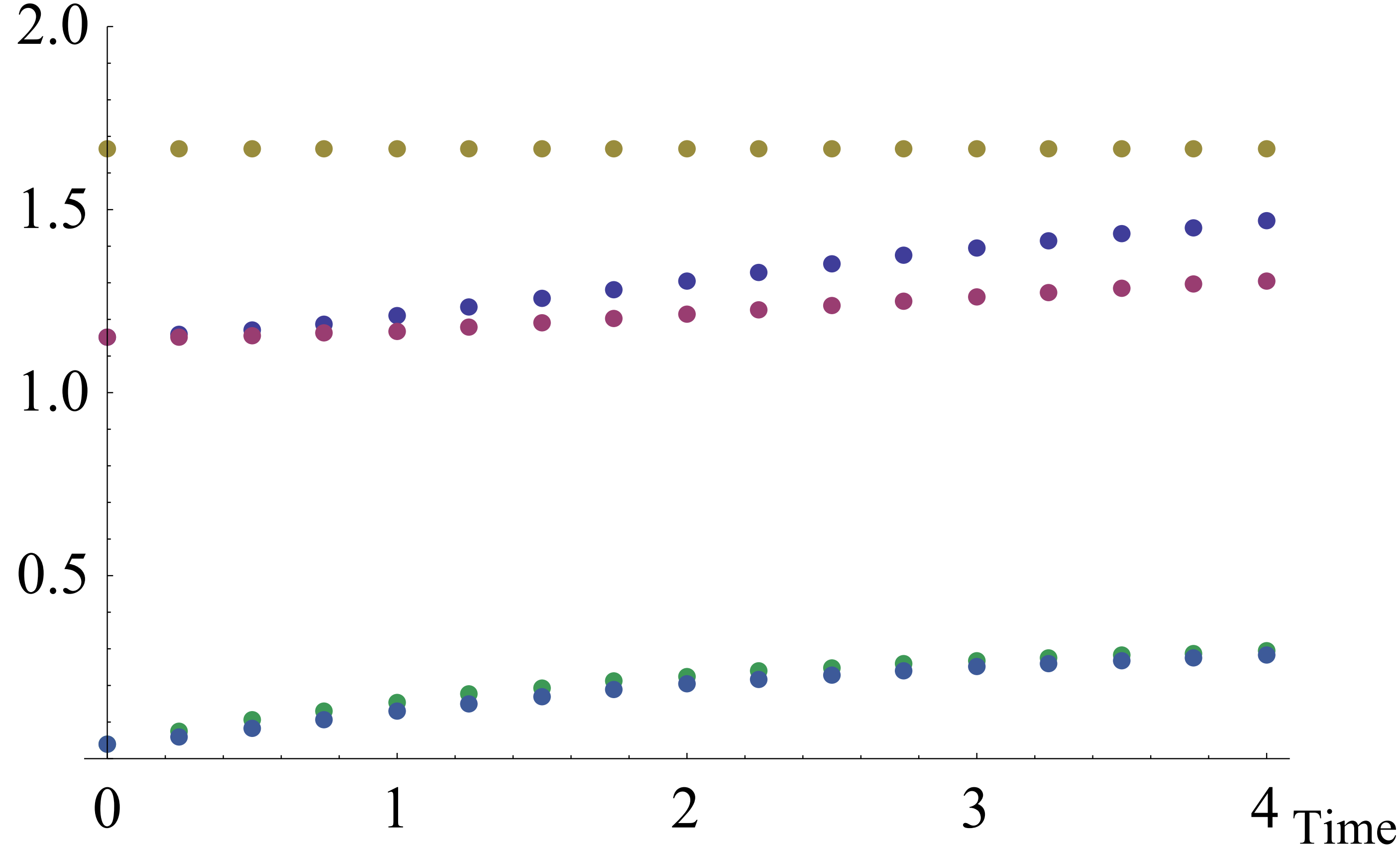


$t = 0 + 1;$



$t = 4$

Comparison of calculated moments with exact ones



$\left\langle v_x^2 \right\rangle$: Blue–calculated, Green–exact; $\left\langle v^4 \right\rangle$: Blue–exact, Red–calculated;

$$\left\langle v^4 \right\rangle_{Maxwellian} = \frac{5}{3}: \text{ Brown}$$

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References

1. V.L. Saveliev, Kinetic Equation for Two-Particle Distribution Function in Boltzmann Gas Mixtures and Equation of Motion for Quasiparticle Pairs. Proc. of the 27th International Symposium on Rarefied Gas Dynamics. Pacific Grove, Ca, USA, 10-15 July 2010, AIP Conference Proceedings (2011), Vol. 1333, pp.134-139
2. V.L.Saveliev, S.A.Filko, S. Yonemura, Kinetic Force Method and Two-particle Kinetic Equation. AIP Conference Proceedings: 29th International Symposium on Rarefied Gas Dynamics, China, X'ian, 13-18 July 2014. –2014
3. Saveliev V.L., and Yonemura S. New forms of two-particle and one-particle kinetic equations// Rarefied Gas Dynamics: 28th International Symposium. AIP Conference Proceedings, vol. 1501, pp. 175-182, (2012)
4. C.Villani, Conservative forms of Bolltzmann's collision operator: Landau revisisited. Math. Mod. An. Num. 33, 1 (1999), 209–227.
5. V.L.Saveliev and K.Nanbu, Collision group and renormalization of the Boltzmann collision integral, Phys. Rev. E 65. 051205. pp.1-9 (2002).
6. Landau,L. D., Zh. Eksp. Teor. Fiz. 7, 203 (1937); see also, Balescu,R., Statistical Mechanics of Charged Particles, Inter-science, New York, 1962.
7. V.L.Saveliev, New Forms of the Boltzmann Collision Integral, Rarefied Gas Dynamics: 25-th International Symposium, edited by M.S.Ivanov and A.K.Rebrov. Novosibirsk 2007

V.L.Saveliev, S.A.Filko, S. Yonemura, Kinetic Force Method and Two-particle Kinetic Equation. AIP Conference Proceedings: 29th International Symposium on Rarefied Gas Dynamics, China, X'ian, 13-18 July 2014. –2014

8. Malkov E.A., Ivanov M.S. Particle-in-Cell method for solving the Boltzmann equation// AIP Conf. Proc. – 2010. – V.1333. – p.940.