Group analysis of the spatially homogeneous and isotropic Boltzmann equation with source using its Fourier image

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JWorkshop in Honor of Michail Ivanov ' 70 th Birthday

June 15-18 2015, Novosibirsk, Russia

The equation studied

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with a source term has the form:

$$\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) = \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\, ds + \hat{q}(x,t).$$

Here the function $\varphi(x,t)$ is

$$\varphi(x,t) \equiv \varphi(k^2/2,t) = \tilde{\varphi}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv) f(v,t)\, dv.$$

where $f(v,t)$ is the distribution function of isotropic in the 3D-space of molecular velocities
Similarly, the transform of the isotropic source function $q(v,t)$ is

$$\tilde{\hat{q}}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv) q(v,t)\, dv,$$

Determining equation

A generator of the admitted Lie group is sought in the form

$$X = \xi(x,t,\varphi)\partial_x + \eta(x,t,\varphi)\partial_t + \zeta(x,t,\varphi)\partial_\varphi.$$

The determining equation for the considered equation is

$$D_t\psi(x,t)+\psi(0,t)\varphi(x,t)+\psi(x,t)\varphi(0,t)-2\int_0^1 \varphi(x(1-s),t)\psi(xs,t)ds = 0,$$

where D_t is the total derivative with respect to t, and the function $\psi(x,t)$ is

$$\psi(x,t) = \zeta(x,t,\varphi(x,t))-\xi(x,t,\varphi(x,t))\varphi_x(x,t)-\eta(x,t,\varphi(x,t))\varphi_t(x,t).$$

Assume that the coefficients of the infinitesimal generator X are represented by the formal Taylor series with respect to φ:

$$\xi(x,t,\varphi) = \sum_{l\geq 0} q_l(x,t)\varphi^l(x,t),$$

$$\eta(x,t,\varphi) = \sum_{l\geq 0} r_l(x,t)\varphi^l(x,t),$$

$$\zeta(x,t,\varphi) = \sum_{l\geq 0} p_l(x,t)\varphi^l(x,t).$$

A particular class of solutions is considered. This class is defined by the initial conditions

$$\varphi_0(x,t) = bx^n$$

at a given (arbitrary) time $t = t_0$. Here, $n = 0, 1, 2, ...$.

The coefficients of the generator X are

$$\xi(x,t,\varphi)=c_0x,\quad \eta(x,t,\varphi)=-c_2t+c_3,\quad \zeta(x,t,\varphi)=(c_2+c_1x)\varphi$$

where $c_0, c_1, c_2$and c_3 are arbitrary constant.
Thus, each admitted generator has the form

$$X=c_0X_0+c_1X_1+c_2X_2+c_3X_3,$$

where

$$X_0=x\partial_x,\quad X_1=x\varphi\partial_\varphi,\quad X_2=\varphi\partial_\varphi-t\partial_t,\quad X_3=\partial_t.$$

The remaining part of the determining equation becomes

$$(c_2t-c_3)\hat{q}_t-c_0x\hat{q}_x+(c_1x+2c_2)\hat{q}=0.$$

Equivalence Transformations

Let us introduce the operator L:

$$L\varphi = \varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,ds.$$

Equivalence transformations corresponding to the generators X_0, X_1, X_2 and X_3, are obtained as follows.
For example, the transformations corresponding to the generator $X_0 = x\partial_x$ map a function $\varphi(x,t)$ into the function $\bar{\varphi}(\bar{x},\bar{t}) = \varphi(\bar{x}e^{-a},\bar{t})$, where a is the group parameter. The transformed expression becomes

$$\begin{aligned}
\bar{L}\bar{\varphi} &= \bar{\varphi}_{\bar{t}}(\bar{x},\bar{t}) + \bar{\varphi}(\bar{x},\bar{t})\bar{\varphi}(0,\bar{t}) - \int_0^1 \bar{\varphi}(\bar{x}s,\bar{t})\bar{\varphi}(\bar{x}(1-s),\bar{t})\,ds \\
&= \varphi_{\bar{t}}(\bar{x}e^{-a},\bar{t}) + \varphi(\bar{x}e^{-a},\bar{t})\varphi(0,\bar{t}) - \int_0^1 \varphi(\bar{x}e^{-a}s,\bar{t})\varphi(\bar{x}e^{-a}(1-s),\bar{t})\,ds \\
&= \varphi_t(x,t) + \varphi(x,t)\varphi(0,t) - \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,ds \\
&= L\varphi.
\end{aligned}$$

This defines the Lie group of equivalence transformations of the equation

$$\bar{x} = xe^a, \quad \bar{t} = t, \quad \bar{\varphi} = \varphi, \quad \bar{\hat{q}} = \hat{q}.$$

Similarly, the transformations corresponding:
to the generator $X_3 = \partial_t$

$$\bar{x} = x, \quad \bar{t} = t + a, \quad \bar{\varphi} = \varphi, \quad \bar{\hat{q}} = \hat{q}.$$

to the generator $X_2 = \varphi\partial_\varphi - t\partial_t$

$$\bar{x} = x, \quad \bar{t} = te^{-a}, \quad \bar{\varphi} = \varphi e^{a}, \quad \bar{\hat{q}} = \hat{q}e^{2a}$$

to the generator $X_1 = x\varphi\partial_\varphi$

$$\bar{x} = x, \quad \bar{t} = t, \quad \bar{\varphi} = \varphi e^{xa}, \quad \bar{\hat{q}} = \hat{q}e^{xa}.$$

Thus the generators defining an equivalence Lie group of the considered equation are

$$X_0^e = x\partial_x, \quad X_1^e = x\varphi\partial_\varphi + x\hat{q}\partial_{\hat{q}}, \quad X_2^e = \varphi\partial_\varphi - t\partial_t + x\hat{q}\partial_{\hat{q}}, \quad X_3^e = \partial_t.$$

Let us study the change of a generator

$$X = x_0 X_0 + x_1 X_1 + x_2 X_2 + x_3 X_3$$

under these equivalence transformations. After the change one gets the generator

$$X = \hat{x}_0 \hat{X}_0 + \hat{x}_1 \hat{X}_1 + \hat{x}_2 \hat{X}_2 + \hat{x}_3 \hat{X}_3,$$

where

$$\hat{X}_0 = \bar{x}\partial_{\bar{x}}, \quad \hat{X}_1 = \bar{x}\bar{\varphi}\partial_{\bar{\varphi}}, \quad \hat{X}_2 = \bar{\varphi}\partial_{\bar{\varphi}} - \bar{t}\partial_{\bar{t}}, \quad \hat{X}_3 = \partial_{\bar{t}}.$$

The corresponding transformations of the basis generators are

$$\begin{aligned}
X_0^{\mathrm{e}}:&\ \ X_0 = \hat{X}_0, X_1 = e^{-a}\hat{X}_1, X_2 = \hat{X}_2, X_3 = \hat{X}_3;\\
X_1^{\mathrm{e}}:&\ \ X_0 = \hat{X}_0 + a\hat{X}_1, X_1 = \hat{X}_1, X_2 = \hat{X}_2, X_3 = \hat{X}_3;\\
X_2^{\mathrm{e}}:&\ \ X_0 = \hat{X}_0, X_1 = \hat{X}_1, X_2 = \hat{X}_2, X_3 = e^{-a}\hat{X}_3;\\
X_3^{\mathrm{e}}:&\ \ X_0 = \hat{X}_0, X_1 = \hat{X}_1, X_2 = \hat{X}_2 + a\hat{X}_3, X_3 = \hat{X}_3.
\end{aligned}$$

Any generator X can be expressed as a linear combination of the basis generators:

$$X = \hat{x}_0 \hat{X}_0 + \hat{x}_1 \hat{X}_1 + \hat{x}_2 \hat{X}_2 + \hat{x}_3 \hat{X}_3 = x_0 X_0 + x_1 X_1 + x_2 X_2 + x_3 X_3$$

Using the invariance of a generator with respect to a change of the variables, the basis generators X_i ($i = 0, 1, 2, 3$) and $\hat{X}_j$ ($j = 0, 1, 2, 3$) in corresponding equivalence transformations are related as follows:

$$\begin{aligned}
X_0^{\mathrm{e}}: &\quad \hat{x}_1 = x_1 e^{-a}, \\
X_1^{\mathrm{e}}: &\quad \hat{x}_1 = x_1 + a x_0, \\
X_2^{\mathrm{e}}: &\quad \hat{x}_3 = x_3 e^{a}, \\
X_3^{\mathrm{e}}: &\quad \hat{x}_3 = x_3 + a x_2.
\end{aligned}$$

For classification an algebraic algorithm was applied,

- first we study the Lie algebra L_4 composed by the generators X_0, X_1, X_2, X_3. The commutator table is

	X_0	X_1	X_2	X_3
X_0	0	X_1	0	0
X_1	$-X_1$	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	X_3	0

The inner automorphisms are

$$\begin{aligned}
A_0 &: \hat{x}_1 = x_1 e^a, \\
A_1 &: \hat{x}_1 = x_1 + a x_0, \\
A_2 &: \hat{x}_3 = x_3 e^a, \\
A_3 &: \hat{x}_3 = x_3 + a x_2,
\end{aligned}$$

- Second, one can notice that the results of using the equivalence transformations corresponding to the generators $X_0^e, X_1^e, X_2^e, X_3^e$ are similar to changing coordinates of a generator X with regards to the basis change.
  These changes are similar to the inner automorphisms.

Really

$$\begin{array}{llll} A_0: \hat{x}_1 = x_1 e^a, & X_0^{\mathrm{e}}: & \hat{x}_1 = x_1 e^{-a}, \\ A_1: \hat{x}_1 = x_1 + a x_0, & X_1^{\mathrm{e}}: & \hat{x}_1 = x_1 + a x_0, \\ A_2: \hat{x}_3 = x_3 e^a, & X_2^{\mathrm{e}}: & \hat{x}_3 = x_3 e^a, \\ A_3: \hat{x}_3 = x_3 + a x_2, & X_3^{\mathrm{e}}: & \hat{x}_3 = x_3 + a x_2. \end{array}$$

Optimal system of subalgebras

Construction of an optimal system of subalgebras of the Lie algebra L_4

- It is simplified if one notices that $L_4 = F_1 \oplus F_2$, where $F_1 = \{X_0, X_1\}$ and $F_2 = \{X_2, X_3\}$ are ideals of the Lie algebra L_4.
- This decomposition gives a possibility to apply a two-step algorithm (Ovsiannikov, 1993 and 1994).

The result of construction of an optimal system of subalgebras is presented in Table 1.

Optimal system of subalgebras

No.	Basis	No.	Basis
1.	$X_0,\ X_1,\ X_2,\ X_3$	13.	$X_0+X_3,\ X_1$
2.	$\gamma X_0+X_2,\ X_1,\ X_3$	14.	$X_1,\ X_3$
3.	$X_0,\ X_1,\ X_3$	15.	$X_0,\ X_3$
4.	$X_0,\ X_1,\ X_2$	16.	$X_0,\ X_1$
5.	$X_0,\ X_2,\ X_3$	17.	γX_0+X_2
6.	$X_2,\ X_3$	18.	X_1+X_2
7.	$X_2+X_0,\ X_1+\ X_3$	19.	X_1-X_2
8.	$X_2+\gamma X_0,\ X_3$	20.	X_0+X_3
9.	$X_1+X_2,\ X_3$	21.	X_1+X_3
10.	$X_1-X_2,\ X_3$	22.	X_0
11.	$X_0,\ X_2$	23.	X_1
12.	$\gamma X_0+X_2,\ X_1$	24.	X_3

Obtaining the Function $\hat{q}$

Example: Lie algebra $\{\gamma X_2 + 2X_0,\ X_3\}$
For this Lie algebra there are two sets of the coefficients c_i, $(i = 0, 1, 2, 3)$:

$$\begin{aligned} \gamma X_2 + 2X_0 : \quad & c_0 = 2 \quad c_1 = 0 \quad c_2 = \gamma \quad c_3 = 0; \\ X_3 : \quad & c_0 = 0 \quad c_1 = 0 \quad c_2 = 0 \quad c_3 = 1. \end{aligned}$$

These sets define the system of equations by substituting the coefficients c_i into the remaining equation:

$$\gamma(\frac{1}{2}t\hat{q}_t + \hat{q}) - x\hat{q}_x = 0, \qquad \hat{q}_t = 0.$$

The general solution of these equations is $\hat{q} = \beta x^{\gamma}$, where β is constant.

Group Classification

No.	$\hat{q}(t,x)$	Generators
1.	0	$X_0,\ X_1,\ X_2\ X_3$
2.	kx^2e^{tx}	$X_2+X_0,\ X_1+X_3$
3.	kx^{γ}	$\gamma X_2+2X_0,\ X_3$
4.	kt^{-2}	$X_0,\ X_2$
5.	$t^{-2}\Phi(xt^{\alpha})$	αX_0+X_2
6.	$t^{-(x+2)}\Phi(x)$	X_1+X_2
7.	$t^{x-2}\Phi(x)$	X_1-X_2
8.	$\Phi(xe^{-t})$	X_0+X_3
9.	$e^{xt}\Phi(x)$	X_1+X_3
10.	$\Phi(t)$	X_0
11.	$\Phi(x)$	X_3

where α,β,γ and k are constant

Representation of invariant solution for $\hat{q} = kx^2 e^{xt}$

Equation

$$\varphi_t(x,t) + \varphi(x,t)\varphi(0,t) = \int_0^1 \varphi(xs,t)\varphi(x(1-s),t)\,ds + kx^2 e^{xt}.$$

The corresponding admitted Lie algebra of the equation is $\{X_2 + X_0,\ X_1 + X_3\}$.
An optimal system of subalgebras of this Lie algebra is:

$$\{X_2 + X_0\},\ \{X_1 + X_3\}, \{X_2 + X_0,\ X_1 + X_3\}$$

For the subalgebra $\{X_2 + X_0\}$ corresponding invariant solution has a representation

$$\varphi = t^{-1} r(z), \quad z = xt$$

Substituting this representation of invariant solution, we obtain the reduced equation:

$$zr'(z) - r(z) + r(z)r(0) - \int_0^1 r(zs)r(z(1-s))\,ds = kz^2 e^z.$$

THANK YOU
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